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ADDENDUM

The classification of Novikov algebras in low dimensions: invariant bilinear forms

Chengming Bai[1,2] **and Daoji Meng**[3]

[1] Theoretical Physics Division, Nankai Institute of Mathematics, Tianjin 300071, People's Republic of China
[2] Liu Hui Center for Applied Mathematics, Tianjin 300071, People's Republic of China
[3] Department of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

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Abstract
In this note, we give a complete classification of the (non-degenerate) symmetric invariant bilinear forms on Novikov algebras in dimension 2 and 3, which can be regarded as an addendum of the classification of Novikov algebras in low dimensions given in our previous work (Bai C M and Meng D J 2001 *J. Phys. A: Math. Gen.* **34** 1581–94).

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Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type [1–4] and Hamiltonian operators in the formal variational calculus [5–8]. We have obtained the classification of the Novikov algebras in low dimensions [9]; there is a kind of realization theory of Novikov algebras [10, 11]. Furthermore, it is important to study the invariant bilinear forms on Novikov algebras. In fact, the (non-degenerate) symmetric invariant bilinear forms on Novikov algebras are related to the pseudo-Riemannian metric and their non-linear changes [3]; they were first studied (mainly on the associative Novikov algebras) in references [3, 4].

Let A be a finite-dimensional Novikov algebra over the base field $\mathbf{F}$ with a bilinear product $(x, y) \to xy$, that is, A satisfies

$$(xy)z - x(yz) = (yx)z - y(xz) \tag{1}$$

$$(xy)z = (xz)y \tag{2}$$

for any $x, y, z \in A$. Let $\{e_1, \ldots, e_n\}$ be a basis of A. A bilinear form $f : A \times A \to \mathbf{F}$ is invariant if and only if

$$f(e_ie_j, e_k) = f(e_i, e_ke_j). \tag{3}$$

If we let

$$f_{ij} = f(e_i, e_j) \tag{4}$$

then the form f under the basis $\{e_1, \ldots, e_n\}$ is completely decided by the matrix $\mathcal{F} = (f_{ij})$. Furthermore, the form is symmetric if and only if $\mathcal{F}$ is symmetric and the form is non-degenerate if and only if the determinant of $\mathcal{F}$ is not zero. All the (symmetric) invariant bilinear forms span a linear space.

Let $\{c_k^{ij}\}$ be the set of structure constants of A, i.e.,

$$e_i e_j = \sum_k^n c_k^{ij} e_k. \tag{5}$$

Then by equation (3), we have

$$\sum_{l=1}^n c_l^{ij} f_{lk} = \sum_{l=1}^n c_l^{kj} f_{il}. \tag{6}$$

This means that f_{ij} can be solved directly through these homogeneous linear equations with the coefficients c_k^{ij}.

In particular, it is interesting to see that the symmetry of the form f is not completely 'independent' with the invariance of f, that is, on some Novikov algebras, the symmetry of the invariant bilinear form is not an additional condition but a consequence of the invariance.

Example. Let us see the invariant bilinear forms on two-dimensional Novikov algebras over the complex number field, for which the classification is given in [9]. Through equation (6) with direct calculation, we have table 1 (we use the same symbols as in [9]).

Table 1.

Type	Invariant bilinear forms	Symmetric invariant bilinear forms	Determinant of symmetric forms
(T1)	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$f_{11}f_{22} - f_{12}^2$
(T2)	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & 0 \end{pmatrix}$	$-f_{12}^2$
(T3)	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$-f_{12}^2$
(N1)	$\mathcal{F} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$	$f_{11}f_{22}$
(N2)	$\mathcal{F} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & 0 \\ 0 & f_{22} \end{pmatrix}$	$f_{11}f_{22}$
(N3)	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & 0 \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & 0 \end{pmatrix}$	$-f_{12}^2$
(N4)	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$f_{11}f_{22} - f_{12}^2$
(N5)	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$-f_{12}^2$
(N6)	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$\mathcal{F} = \begin{pmatrix} 0 & f_{12} \\ f_{12} & f_{22} \end{pmatrix}$	$-f_{12}^2$

So we know that except (T1) and (N4) the symmetry of invariant bilinear forms on two-dimensional Novikov algebras are completely decided by the invariance. We also would like to point out that the case for (T3) is an example given by Zel'manov in [4].

Next we give all the symmetric invariant bilinear forms on three-dimensional Novikov algebras over the complex number field in table 2 through equation (6) with direct calculation (we use the same symbols as in [9]):

Table 2.

Type	Symmetric invariant bilinear forms	$\det\mathcal{F}$	Type	Symmetric invariant bilinear forms	$\det\mathcal{F}$
(A1)	$\mathcal{F}=\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{12} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$\det\mathcal{F}$	(A2)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(A3)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{23} & f_{33} \end{pmatrix}$	0	(A4)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{22} \\ 0 & f_{22} & f_{23} \\ f_{22} & f_{23} & f_{33} \end{pmatrix}$	$-f_{22}^3$
(A5)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{23} & f_{33} \end{pmatrix}$	0	(A6)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & f_{22} & f_{23} \\ 0 & f_{23} & f_{33} \end{pmatrix}$	0
(A7)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{22} \\ 0 & f_{22} & f_{23} \\ f_{22} & f_{23} & f_{33} \end{pmatrix}$	$-f_{22}^3$	(A8)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0
(A9)	$\mathcal{F}=\begin{pmatrix} f_{11} & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$-f_{11}f_{23}^2$	(A10)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0
(A11)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0	(A12)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0
(A13)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 2f_{13} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$-2f_{13}^3$			
(B1)	$\mathcal{F}=\begin{pmatrix} f_{11} & 0 & 0 \\ 0 & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$	$f_{11}f_{22}f_{33}$	(B2)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(B3)	$\mathcal{F}=\begin{pmatrix} f_{11} & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$f_{22}\left(f_{11}f_{33}-f_{13}^2\right)$	(B4)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(B5)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$			
(C1)	$\mathcal{F}=\begin{pmatrix} f_{11} & f_{12} & 0 \\ f_{12} & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}$	$f_{33}\left(f_{11}f_{22}-f_{12}^2\right)$	(C2)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(C3)	$\mathcal{F}=\begin{pmatrix} f_{11} & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$f_{22}\left(f_{11}f_{33}-f_{13}^2\right)$	(C4)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(C5)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & 0 \\ f_{13} & 0 & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$	(C6)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$
(C7)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0	(C8)	$\mathcal{F}=\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$\det\mathcal{F}$
(C9)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & f_{22} & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	$-f_{22}f_{13}^2$	(C10)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0
(C11)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0	(C12)	$\mathcal{F}=\begin{pmatrix} 0 & 0 & f_{13} \\ 0 & 0 & f_{23} \\ f_{13} & f_{23} & f_{33} \end{pmatrix}$	0

Table 2. (*Continued*)

Type	Symmetric invariant bilinear forms	det $\mathcal{F}$	Type	Symmetric invariant bilinear forms	det $\mathcal{F}$
(C13)	$\mathcal{F}=\begin{pmatrix}0&0&f_{13}\\0&0&f_{23}\\f_{13}&f_{23}&f_{33}\end{pmatrix}$	0	(C14)	$\mathcal{F}=\begin{pmatrix}0&0&f_{13}\\0&0&f_{23}\\f_{13}&f_{23}&f_{33}\end{pmatrix}$	0
(C15)	$\mathcal{F}=\begin{pmatrix}0&0&f_{13}\\0&0&f_{23}\\f_{13}&f_{23}&f_{33}\end{pmatrix}$	0	(C16)	$\mathcal{F}=\begin{pmatrix}0&0&f_{13}\\0&f_{22}&f_{23}\\f_{13}&f_{23}&f_{33}\end{pmatrix}$	$-f_{22}f_{13}^2$
(C17)	$\mathcal{F}=\begin{pmatrix}0&0&f_{13}\\0&0&f_{23}\\f_{13}&f_{23}&f_{33}\end{pmatrix}$	0	(C18)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&0\\f_{13}&0&f_{33}\end{pmatrix}$	0
(C19)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&0\\f_{13}&0&f_{33}\end{pmatrix}$	0			
(D1)	$\mathcal{F}=\begin{pmatrix}f_{11}&f_{12}&0\\f_{12}&0&0\\0&0&f_{33}\end{pmatrix}$	$-f_{33}f_{12}^2$	(D2)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&f_{11}\\f_{13}&f_{11}&f_{33}\end{pmatrix}$	$-f_{11}^3$
(D3)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&f_{11}\\f_{13}&f_{11}&f_{33}\end{pmatrix}$	$-f_{11}^3$	(D4)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&\frac{1}{2}f_{11}\\f_{13}&\frac{1}{2}f_{11}&f_{33}\end{pmatrix}$	$-\frac{1}{4}f_{11}^3$
(D5)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&\frac{1}{2}f_{11}\\f_{13}&\frac{1}{2}f_{11}&f_{33}\end{pmatrix}$	$-\frac{1}{4}f_{11}^3$	(D6)	$\mathcal{F}=\begin{pmatrix}f_{11}&0&f_{13}\\0&0&lf_{11}\\f_{13}&lf_{11}&f_{33}\end{pmatrix}$	$-l^2f_{11}^3$
(E1)	$\mathcal{F}=\begin{pmatrix}0&f_{12}&0\\f_{12}&f_{22}&0\\0&0&f_{33}\end{pmatrix}$	$-f_{33}f_{12}^2$			

At the end of this note, we give the following conclusion from the above discussion:

(a) There exist non-degenerate symmetric invariant bilinear forms on any two-dimensional Novikov algebras. There is no non-degenerate symmetric invariant bilinear form on some three-dimensional Novikov algebras no matter whether they are associative or not [4].

(b) We can see that many (non-isomorphic) Novikov algebras have the same (non-degenerate) symmetric bilinear forms. One of the reasons is perhaps due to their close relations with the commutative associative algebras through the realization theory [10, 11].

(c) We would like to point out that the symmetric invariant bilinear forms on (T1), (A1), (N4) and (C8) are a little 'special': any symmetric bilinear form is invariant. Note that (C8) just corresponds to the Poisson brackets of one-dimensional hydrodynamics, which was already discussed in [3].

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